Lecture 20

Anti Derivatives

A function F is called an antiderivative of f an interval I

if $F'(x) = f(x)$. for all x in I .

Let $f(x) = 3x^2$

Then $F(x) = x^3$, $f'(x) = 3x^2$. So F is an antiderivative of f .

But $G(x) = x^3 + 1$ also has derivative $G'(x) = 3x^2$.

So F and G are both antiderivatives of f .

Actually in general, any function $H(x) = x^3 + C$, where C is

a constant , is an antiderivative of f .

Are there any other function

Recall from MVT class , we showed that if

two functions have the same derivative on an interval , then they

must differ by a constant.

Thus if F and G are two antiderivatives of f , then $F'(x) = G'(x) = f(x)$

so, $G(x) = F(x) + C$, where C is a constant .

Thm If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x)+C$ where C is an arbitrary constant

Ex Find the most general antiderivative of the following functions:

a) $f(x) = \cos x$

If $F(x) = \sin x$, then $F'(x) = \cos x$

So an antiderivative of $\cos x$ is $\sin x$

Then the most general anti derivative is $F(x) = \sin x + C$.

b) $f(x) = x^n$, $n \geqslant 0$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{(n+1)} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$, because $f(x) = x^n$ is defined on any interval.

c) $f(x) = x^{-3}$

If we put $n = -3$ in part (b), we get a particular derivative

$$F(x) = \frac{x^{-2}}{(-2)}$$ by the same calculation.

But $f(x)$ is not defined at $x = 0$

Thus Thm 1, tells us that that the only general antiderivative of

f is $\frac{x^{-2}}{-2} + C$ on any interval that doesn't contain 0.

So general antiderivative is

$$F(x) = \begin{cases} -\frac{1}{2x^2} + C_1, & x > 0 \\ -\frac{1}{2x^2} + C_2, & x < 0 \end{cases}$$

Table of Antidifferentiation formulas.

let $F' = f$, $G' = g$

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$

Ex Find the most general antiderivative of

$$f(x) = \frac{2\sqrt{x^3} + \sqrt[4]{x^5}}{\sqrt[6]{x}}$$

Soln

$$\frac{2x^{3/2} + x^{5/4}}{x^{1/6}} = \frac{2x^{3/2}}{x^{1/6}} + \frac{x^{5/4}}{x^{1/6}}$$

$$= 2x^{3/2 - 1/6} + x^{5/4 - 1/6}$$

$$= 2x^{4/3} + x^{13/12}$$

Then general antiderivative is

$$\frac{2x^{4/3+1}}{\frac{4}{3}+1} + \frac{x^{\frac{13}{12}+1}}{\frac{13}{12}+1} + C$$

$$= \frac{2\,x^{7/3}}{7/3} + \frac{x^{25/12}}{25/12} + C$$

$$= \frac{6}{7}x^{7/3} + \frac{12}{25}x^{25/12} + C$$

Example Find f if $f'(x) = x\sqrt{x}$ and if $f(1) = 2$

Then, $f'(x) = x^{3/2}$. To find $f(x)$, we need to find the general antiderivative.

So $f(x) = \dfrac{x^{3/2+1}}{\frac{3}{2}+1} + C = \dfrac{2}{5}x^{5/2} + C$

To determine C we use the fact that $f(1) = 2$

So, $f(1) = \frac{2}{5}(1)^{5/2} + C = 2 \Rightarrow C = 2 - \frac{2}{5} = \frac{8}{5}$

So, $f(x) = \frac{2}{5}x^{5/2} + \frac{8}{5}$

So extra condition uniquely determines the antiderivative

Ex $f''(x) = x^{-3}$, $x > 0$, $f(1) = 0$, $f(2) = 0$. Find $f(x)$

Then $f'(x)$ is the anti derivative of $f''(x)$.

So $f'(x) = \dfrac{x^{-3+1}}{-3+1} + C = \dfrac{x^{-2}}{-2} + C$ (For $x > 0$)

Then finding ~~an~~ $f(x)$ is finding anti derivative of $f'(x)$.

$$f(x) = -\frac{1}{2}\,\frac{x^{-2+1}}{-2+1} + Cx + D$$

$$= -\frac{1}{2}\,\frac{x^{-1}}{-1} + Cx + D$$

$$= \frac{1}{2x} + Cx + D$$

Now $f(1) = \dfrac{1}{2} + C + D = 0 \Rightarrow C + D = -\dfrac{1}{2}$

$f(2) = \dfrac{1}{4} + 2C + D = 0 \Rightarrow 2C + D = -\dfrac{1}{4}$

$C = \dfrac{1}{4}$, $D = -\dfrac{3}{4}$

So, $f(x) = \dfrac{1}{2x} + \dfrac{1}{4}x - \dfrac{3}{4}$

Lec 20

Recall $s'(t) = v(t)$

$v'(t) = a(t) \Rightarrow s''(t) = a(t)$

Ex A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. It's initial velocity is $v(0) = -6$ cm/s and it's initial displacement $s(0) = 9$ cm. Find the position function $s(t)$.

Soln $a(t) = v'(t) = 6t + 4$

Then general antiderivative is $v(t) = \frac{6t^2}{2} + 4t + C = 3t^2 + 4t + C$

As $v(0) = -6$, $v(0) = C \Rightarrow -6 = C \Rightarrow v(t) = 3t^2 + 4t - 6$

Next $s'(t) = v(t) = 3t^2 + 4t - 6$

Then $s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$

As $s(0) = 9$, $s(0) = D$, $D = 9$

Hence, $s(t) = t^3 + 2t^2 - 6t + 9$

Ex

Suppose that the marginal cost at a production level of x units per day of some product is given by $C'(x) = 0.03x^2 - 1.2x + 300$ dollars per unit and total production costs at $x = 300$ units per day is \$ 500,000. What is the daily fixed costs of production? How

Sol^n Marginal cost is the derivative of cost function so $C(x)$ is the antiderivative of marginal cost.

Then $$C(X) = 0.03 \cdot \frac{x^{2+1}}{2+1} - \frac{1.2\, x^{1+1}}{2} + 300x + D$$

$$= 0.01x^3 - 0.6x^2 + 300x + D.$$

From the question, $C(300) = 500,000$

$$(0.01)(300)^3 - 0.6(300)^2 + 300(300) + D = 500,000$$

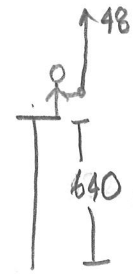
$$270,000 + (300)^2(.4) + D = 500,000$$

$$D = 500,000 - 306,000$$

$$D = 194,000$$

So, $C(x) = 0.01x^3 - 0.6x^2 + 300 + 194,000$.

Ex A ball is thrown upward with a speed of 48 ft/s, from the edge of a cliff 640 ft above the ground. Find its height above the ground t secs later. When does it reach its maximum height? When does it hit the ground?



Soln Motion is vertical. Choose positive direction upward

At time t the distance above the ground is $s(t)$ and velocity $v(t)$ is decreasing. So acceleration is negative and we have

$a(t) = -32 \text{ ft}/s^2$.

Then $a(t) = v'(t)$, so antiderivative is $v(t) = -32t + D$

To determine D we use the fact that $v(0) = 48$.

So, $v(0) = 48 = D \Rightarrow v(t) = -32t + 48$

The maximum ht is reached when $v(t) = 0 \Rightarrow t = 1.5s$.

So, $\cancel{s}v(t) = s'(t) = -32t + 48$

Again, take antiderivative, we get $s(t) = -16t^2 + 48t + C$

and $s(0) = 640 \Rightarrow C = 640$

So, $s(t) = -16t^2 + 48t + 640$

To find when the ball hits the ground, $s(t) = 0$

$\Rightarrow -16t^2 + 48t + 640 = 0$

$\Rightarrow -16(t^2 - 3t - 40) = 0$

$\Rightarrow (t+5)(t-8) = 0 \Rightarrow t = -5, 8$

Reject negative, so $t = 8s$.